Light meson decays

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Light mesons decay via weak interactions. Their study will allow us to better understand how EW interactions are constrained by these low energy processes, in particular the flavor structure as dictated by the CKM.

The charged current interactions are given by

$$\mathcal{L}^{c.c.} = \frac{g}{2\sqrt{2}} W^{+}_{\mu} J^{-\mu} + h.c.$$

$$; \ J^{-\mu} = (V_{CKM})_i{}^j \, \bar{u}^i \gamma^\mu (1-\gamma^5) d_j$$
$$+ \bar{\nu}_e \gamma^\mu (1-\gamma^5) e + \bar{\nu}_\mu \gamma^\mu (1-\gamma^5) \mu$$
$$\equiv J^{-\mu}_{had} + J^{-\mu}_{lep}$$

We also saw that, to a good approx,

$$V_{CKM} = \begin{pmatrix} \cos\theta_c & \sin\theta_c & 0 \\ -\sin\theta_c & \cos\theta_c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the charged hadronic current is

$$J^{-\mu}_{had} = \cos\theta_c \, \bar{u}\gamma^\mu(1-\gamma^5) d + \sin\theta_c \, \bar{u}\gamma^\mu(1-\gamma^5) s$$

At low energies, the current enters in the Fermi theory as

$$\mathcal{L}^F = -\frac{G_F}{\sqrt{2}} J^+_\mu J^{-\mu} - \frac{4 G_F}{\sqrt{2}} J^0_\mu J^{0\mu}$$

. Since QCD interactions are nonperturbative, writing the full lagrangian as

$$\mathcal{L} = \mathcal{L}^{QCD} + \mathcal{L}^{Fermi} + \mathcal{L}^{QED}$$

is not very useful. But we will see that in some cases one can factorize the correlator in a perturbative times a non-perturb. part.

Take the example of

$$\pi^+ \to \ell^+ \nu_e$$

This is necessarily mediated by the weak force. Under QCD, π^+ would be stable.

We want to compute the width, i.e.
the matrix element

$$\langle \ell^+ \nu_\ell | S | \pi^+ \rangle$$

we can extract it from the correlator

$$\langle 0 | T \{ \bar{\Psi}_e(x_e) \Psi_\nu(x_\nu) \mathcal{O}_\pi^\dagger(x_\pi) \} | 0 \rangle$$

where $\mathcal{O}_\pi$ is a local operator interpolating pion states.

The path integral is

$$\int Dq\, D\bar{q}\, DG\, D(\text{lept.})\ \bar{\Psi}_e \Psi_\nu \mathcal{O}_\pi^\dagger e^{i(S_{QCD} + S_{lep} + S_{Fermi})}$$

$$= \int D(\text{lep}) \bar{\Psi}_e \Psi_\nu e^{iS_{lep}} \int Dq\, D\bar{q}\, DG\ \mathcal{O}_\pi^\dagger e^{iS_{QCD} + iS_{Fermi}}$$

we can do, at least formally, the QCD part & then the leptonic part.

Fermi term mixes both, but we are interested in the pert. expansion,

$$e^{iS_{Fermi}} = 1 + iS_{Fermi} + \dots$$

$$= 1 + i \int d^4x \left(-\frac{G_F}{\sqrt{2}} J_\mu^+ J^{-\mu} \right) + \dots$$

The first term "1" gives zero in the PI.

Writing

$$J_\mu^+ J^{-\mu} = J_{had}^+ \cdot J_{lep}^- + J_{lep}^+ \cdot J_{lep}^- + J_{had}^+ \cdot J_{had}^-$$

we see that only the first term is relevant for the decay. So the correlator is

$$\langle 0 | T\{ \bar{\psi}_e(x_e) \psi_\nu(x_\nu) \mathcal{O}_\pi^\dagger(x_\pi) \} | 0 \rangle \simeq$$

$$\simeq -\frac{iG_F}{\sqrt{2}} \int d^4x \, \langle 0 | T\{ \bar{\psi}_e(x_e) \psi_\nu(x_\nu) J_{lep,\mu}^-(x) \} | 0 \rangle \times$$

$$\times \langle 0 | T\{ J_{had}^{+,\mu}(x) \, \mathcal{O}_\pi^\dagger(x_\pi) \} | 0 \rangle$$

The hadronic term is evaluated with S_{QCD}, while the others with S_{QED}, $S_{lept.}$

The S-matrix element is then

$$\langle \ell^+ \nu | S | \pi^+ \rangle \equiv i\mathcal{A}(\pi^+ \to \ell^+ \nu_\ell)\, (2\pi)^4 \delta^4(p_{in} - p_{out})$$

$$= -\frac{iG_F}{\sqrt{2}} \int d^4x \, \Big(\langle \ell^+ \nu | J_{lep}^-(0) | 0 \rangle \Big)_{free} \times$$

$$\times \Big(\langle 0 | J_{had}^{+\mu}(x) | \pi^+ \rangle \Big)_{QCD}$$

Using $\mathcal{O}(x) = e^{ip\cdot x}\, \mathcal{O}(0)\, e^{-ip\cdot x}$, one can write the integral over x as

$$\int d^4x\, e^{i(p_{out}-p_{in})\cdot x} = (2\pi)^4 \delta^4(p_{out}-p_{in})$$

Therefore the amplitude is the product of two "form factors",

$$A(\pi^+ \to \ell^+ \nu_\ell) = -\frac{G_F}{\sqrt{2}} L^\mu H_\mu$$

with

$$L^\mu = \langle \ell^+ \nu_\ell | J^{-\mu}_{lep}(0) | 0 \rangle_{Free}$$

$$H^\mu = \langle 0 | J^{+\mu}_{had}(0) | \pi^+ \rangle_{QCD}$$

This factorized stucture is general for semileptonic decays of mesons due to weak interactions.

- The leptonic tensor is just

$$L^\mu = \bar{u}(p_\nu) \gamma^\mu (1-\gamma^5) v(p_e)$$

- For the hadronic tensor, we use symmetries. Since the pion state has no strangeness S, we have

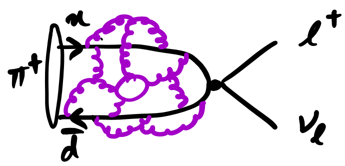
$$H^\mu = \langle 0 | J^{+\mu}_{had}(0) | \pi^+ \rangle$$

$$= \cos\theta_c \langle 0 | \bar{d} \gamma^\mu (1-\gamma^5) u | \pi^+ \rangle$$

$$+ \sin\theta_c \langle 0 | \cancel{\bar{s} \gamma^\mu (1-\gamma^5) u | \pi^+ \rangle}^{\,0}$$

$$\equiv \cos\theta_c \tilde{H}^\mu$$

This could be guessed from a Feyn. diagram



$$= -\frac{G_F}{\sqrt{2}} \cos\theta_c L^\mu \tilde{H}_\mu$$

with $\tilde{H}^\mu = \langle 0 | \bar{d} \gamma^\mu (1-\gamma^5) u | \pi^+ \rangle$.

Lorentz symmetry implies that

$$\tilde{H}^\mu(p) = c(p^2) p^\mu$$

but at $p^2 = m_\pi^2$, $c(p^2)$ is just a constant,

so

$$\tilde{H}^\mu(p) = -i f_\pi p^\mu.$$

This is indeed the pion decay constant.

One can see it from

$$\tilde{H}^\mu = \langle 0 | \bar{d} \gamma^\mu u | \pi^+ \rangle - \langle 0 | \bar{d} \gamma^\mu \gamma^5 u | \pi^+ \rangle$$

So there is the vector & axial part.

Given that $|\pi^+\rangle$ is pseudoscalar, and that $\hat{A}^\mu$ must be a vector, the only nonvanishing part is

$$\hat{A}^\mu = -\langle 0|\bar{d}\gamma^\mu\gamma^5 u|\pi^+\rangle$$
$$= -i f_\pi p^\mu$$

- In summary, the amplitude is

$$\mathcal{A}(\pi^+ \to \ell^+\nu_\ell) = i\frac{G_F}{\sqrt{2}}\cos\theta_c f_\pi \,\bar{u}(p_\nu)\gamma^\mu(1-\gamma^5)v(p_\ell)p_\mu$$

  - G_F is obtained from μ decay
  - $\cos\theta_c$ is a fundamental parameter we want to determine
  - f_π is a "phenomenological" parameter, calculable in principle within QCD.

  Using lattice to do the path integral, one gets

$$f_\pi = 130.2 \text{ MeV} \quad (\pm 1.7 \text{ MeV})$$

  We can also determine f_π from another measurement.

- Using momentum conservation & Dirac eq., the amplitude can be written as

$$\mathcal{A}(\pi^+ \to \ell^+ \nu_\ell) = -i \frac{G_F}{\sqrt{2}} \cos\theta_c \, f_\pi \, m_\ell \, \bar{u}(p_\nu)(1+\gamma^5) v(p_\ell)$$

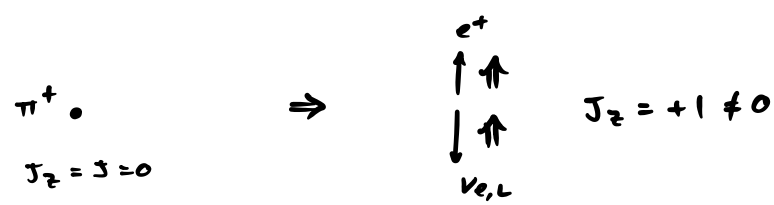
so the amp. is proportional to lepton mass.

The final result is

$$\Gamma(\pi^+ \to \ell^+ \nu_\ell) = \underbrace{\frac{m_\pi}{8\pi}}_{\text{estimate}} \underbrace{\left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2}_{\text{phase space}} \underbrace{\left(G_F^2 f_\pi^2 \cos^2\theta_c \, m_\pi^2 \right)}_{\text{dimensionless coupling}} \underbrace{\frac{m_\ell^2}{m_\pi^2}}_{\text{suppression factor}}$$

The suppression factor comes from the need of a "chirality flip".

Since the leptons couple to a vector current but the pion is a scalar, so



The decay requires a "chirality flip", since the wrong chirality component of an helicity eigenstate is suppressed by the mass.

- From these, one gets

$$\frac{\Gamma(\pi^+ \to e^+ \nu_e)}{\Gamma(\pi^+ \to \mu^+ \nu_\mu)} = \left(\frac{m_e}{m_\mu}\right)^2 \frac{\left(1 - m_e^2/m_\pi^2\right)^2}{\left(1 - m_\mu^2/m_\pi^2\right)^2} \simeq 1.28 \cdot 10^{-4}$$

experiment:
$$BR(\mu^+ \nu_\mu) = 99.9877\%$$
$$BR(e^+ \nu_e) = 1.230 \cdot 10^{-4}$$

- Using the lattice value of f_π, one can extract $\cos\theta_c$.

- The $\cos\theta_c$ can also be accessed via

$$K^+ \to \ell^+ \nu_\ell$$

which is a very similar decay, but with $f_\pi \to f_K$ & $\cos\theta_c \to \sin\theta_c$.

Assuming SU(3) chiral, $f_\pi = f_K$ and one can get θ_c from the π^+ to K^+ leptonic decays, and extract $\frac{\sin\theta_c}{\cos\theta_c}$ from $\frac{\Gamma(K^+ \to e\nu)}{\Gamma(\pi^+ \to e\nu)}$

- The last decay to be discussed is the semileptonic 3-body process

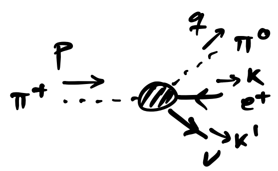
$$\pi^+ \to \pi^0 e^+ \nu_e$$

that gives an alternative determination of $\cos\theta_c$.

- We can write the amplitude as

$$A(\pi^+ \to \pi^0 e^+ \nu_e) = -\frac{G_F}{\sqrt{2}} \cos\theta_c \, L^\mu H_\mu$$

using the same type of manipulations as for the π^+ leptonic decay.



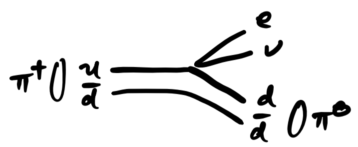
the leptonic matrix element is the same as before,

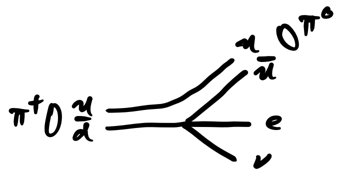
$$L^\mu = \bar{u}(k') \gamma^\mu (1-\gamma^5) v(k)$$

while the hadronic matrix element is

$$H^\mu = \langle \pi^0(q) | (\bar{d} \gamma^\mu (1-\gamma^5) u)(x=0) | \pi^+(p) \rangle$$

Graphically, this comes from





Now the form factor depends on both q^μ and p^μ. Thus, we can write it as

$$H^\mu = C(u^2)(p+q)^\mu + D(u^2)\, u^\mu$$

where we have defined $u^\mu \equiv p^\mu - q^\mu$, the transfer momentum.

So H^μ is some linear combination of $p+q$ and u (or p and q). The coefficients C & D depend on a single invariant, chosen to be u^2. Note that $p^2 = m_{\pi^+}^2$ and $q^2 = m_{\pi^0}^2$.

· Next, we impose parity. Both π^+, π^0 are odd, while H^μ is a vector, so we reach to the conclusion that

$$\begin{aligned} H^\mu &= \langle \pi^0(q) | \bar d \gamma^\mu u | \pi^+(p) \rangle \\ &= C(u^2)(p+q)^\mu + D(u^2)\, u^\mu . \end{aligned}$$

But $\bar{d}\gamma^\mu u$ is the current of an unbroken global symmetry, $SU(2)_I$. The argument applies to the isospin symmetry, and it is insensitive to the larger $SU(3)_V$ and the breaking due to the strange mass. So the prediction is more accurate than the $f_K = f_\pi$ relation.

The currents of the chiral $SU(3)_L \times SU(3)_R$, restricted to $A = a = 1, 2, 3$, are

$$J^a_{L,\mu} = \frac{1}{2}\bar{q}\gamma^\mu(1-\gamma^5)\tau^a q$$
$$J^a_{R,\mu} = \frac{1}{2}\bar{q}\gamma^\mu(1+\gamma^5)\tau^a q$$

so $J^a_{V,\mu} = \bar{q}\gamma^\mu\tau^a q$ is the vector associated with the vector $SU(2)_I$ subgroup.

In particular,

$$\bar{d}\gamma^\mu u = J_V^{\mu,1} - iJ_V^{\mu,2} = \bar{q}\gamma^\mu \underbrace{\tau^-}_{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} q$$

so the vector current that appears in

the matrix element is a linear combination of the isospin currents.

- We will be able to use a Ward identity to simplify our form factor H^μ.

You've probably seen the derivation in QFT III, but let's review it.

Consider a local redefinition

$$\vec{\phi} \to \Omega_\phi(x)\,\vec{\phi} = e^{i\alpha_A(x)T^A}\phi(x)$$

and consider $\alpha_A(x)$ s.t. $\alpha_A(x) \to 0$ for $x \to \infty$.

This does not change the boundary conditions.

Under an infinitesimal transformation, the action transforms as

$$S[\vec{\phi} + i\alpha_A T^A \vec{\phi}] = \int d^4x\, \alpha_A(x)\, \partial^\mu J^A_\mu(x)$$

by the definition of the current. Then, the correlation functions are related by

$$\int D\vec{\phi}\; \mathcal{O}^{a_1}(x_1) \cdots \mathcal{O}^{a_n}(x_n)\, e^{iS[\vec{\phi}]} =$$

$$= \int D\vec{\phi}\, \left[\mathcal{O}^{a_1}(x_1) + i\alpha_A(x_1)(T^A)^{a_1}{}_{b_1}\, \mathcal{O}^{b_1}(x_1)\right] \cdots [\;n\;] \times$$

$$\times\, e^{iS[\vec{\phi}]} \times \left(1 + i\int d^4x\, \alpha_A(x)\, \partial^\mu J^A_\mu(x)\right)$$

where the transformation of each $\mathcal{O}$ depends on their representation.

Consider the limit where $\alpha_A(x) = \delta_{A\bar{A}}\,\delta(x-\bar{x})$.

We obtain the Ward identity

$$\langle 0 | T\{ \partial_\mu J^\mu_A(\bar{x})\, \mathcal{O}^{a_1}(x_1) \dots \mathcal{O}^{a_n}(x_n) \} | 0 \rangle =$$

$$= -\sum_{i=1}^{n} \delta^4(x_i-\bar{x})\, (T^{\bar{A}})^{a_i}{}_{b_i}\, \langle 0 | T\{ \mathcal{O}^{a_1}(x_1) \dots \mathcal{O}^{b_i}(x_i) \dots \mathcal{O}^{a_n}(x_n) \} | 0 \rangle$$

- We want to compute the matrix element of $\partial_\mu J^\mu_A$ between two single-particle states. This is proportional to

$$\langle 0 | T\{ \partial^\mu J_\mu(\bar{x})\, \mathcal{O}^{OUT}(x_{OUT}) (\mathcal{O}^{IN}(x_{IN}))^\dagger \} | 0 \rangle$$

in the limit $x_{IN} \to -\infty$ and $x_{OUT} \to \infty$. This is proportional to a sum of $\delta(x_{IN}-\bar{x})$ and $\delta(x_{OUT}-\bar{x})$ terms, which is zero for $x_{IN/OUT} \to \pm\infty$ while keeping $\bar{x}$ constant.

Therefore, we conclude that

$$\langle \text{1-part, out} | \partial_\mu J^\mu_A(x) | \text{1-part, IN} \rangle = 0$$

This is the same conclusion if we naively applied the classical $\partial_\mu J^\mu = 0$ relation.

· Going back to the matrix element, we find

$$\langle \pi^0(q) | (\bar{d}\gamma^\mu(1-\gamma^5)u)(\bar{x}) | \pi^+(p) \rangle =$$

$$= e^{i\underbrace{(q-p)}_{u^\mu}\cdot\bar{x}} \langle \pi^0(q) | (\bar{d}\gamma^\mu(1-\gamma^5)u)(x=0) | \pi^+(p) \rangle$$

so taking $\partial/\partial\bar{x}^\mu$,

$$u_\mu H^\mu = 0.$$

Therefore,

$$0 = C(u^2)\underbrace{u_\mu(p+q)^\mu}_{=p^2-q^2 = m_{\pi^0}^2 - m_{\pi^+}^2 = 0} + D(u^2)u^2$$

since we are deriving the sum rules imposing isospin, it would be meaningless to consider $m_{\pi^0}^2 \neq m_{\pi^+}^2$. We therefore get $D=0$,

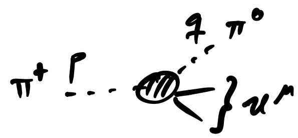
$$H^\mu = C(u^2)(p+q)^\mu$$

· We want the value of c at small k^2.

Since

$$\left.\begin{aligned} m_{\pi^+} &= 139.57 \text{ MeV} \\ m_{\pi^0} &= 134.98 \text{ MeV} \end{aligned}\right\} \delta m_\pi = m_{\pi^+} - m_{\pi^0} \sim 5 \text{ MeV} \ll \Lambda_{QCD}$$

we have that, kinematically,



$$\longrightarrow \quad m_{\pi^+} > m_{\pi^0} + \sqrt{k^2}$$

and

$$\sqrt{k^2} < \delta m_\pi \ll \Lambda_{QCD}$$

So the decay only happens for small $k^2 \sim 0$.

· We can compute $c(0)$ from the following. Take $J^\mu_A(x)$ to be a current operator. Its associated charge

$$Q^A = \int d^3x \, J^0_A(\vec{x}, 0)$$

acts on single particle states as

$$Q^A |\psi^a(\vec{p})\rangle = (T^A)_b{}^a |\psi^b(\vec{p})\rangle$$

Moreover, we have that

$$\langle \Psi_b(q') | Q^A | \Psi^a(p) \rangle = (2\pi)^3\, 2E_{\vec{p}}\, \delta^3(\vec{p}-\vec{q}')\, (T^A)_b{}^a$$

Suppose you want to compute

$$\langle \Psi_b(q) | J_A^\mu(x) | \Psi^a(p) \rangle =$$

$$= e^{iux} \langle \Psi_b(q) | J_A^\mu(0) | \Psi^a(p) \rangle$$

$$\equiv e^{iux} (H_A^\mu)_b{}^a$$

We can compute the $\int d^3x$ of the $\mu=0$ component,

$$(H_A^0(p,q))_b{}^a \int d^3x\, e^{-i\vec{u}\cdot\vec{x}} = (2\pi)^3 \delta^3(\vec{u}) (H_A^0(p,q))_b{}^a$$

while from the matrix element of Q one gets

$$(2\pi)^3 \delta^3(\vec{u})\, 2E_{\vec{p}}\, (T^A)_b{}^a$$

So we get the relation

$$(H_A^0(p,q)|_{p=q})_b{}^a = 2E_{\vec{p}}\, (T^A)_b{}^a$$

in the zero-momentum limit. So the matrix element is fully determined by symmetry.

By applying the general relation to the case of $\pi^+ \to \pi^0 e^+ \nu_e$, one finds

$$c(0) = -\sqrt{2}.$$

So the $\pi^+ \to \pi^0 e^+ \nu_e$ partial width is written exclusively in terms of the Cabbibo angle.